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# EXTENDED SPECTRUM AND EXTENDED EIGENSPACES OF QUASI-NORMAL OPERATORS

GILLES CASSIER AND HASAN ALKANJO

ABSTRACT. We say that a complex number  $\lambda$  is an extended eigenvalue of a bounded linear operator  $T$  on a Hilbert space  $\mathcal{H}$  if there exists a nonzero bounded linear operator  $X$  acting on  $\mathcal{H}$ , called extended eigenvector associated to  $\lambda$ , and satisfying the equation  $TX = \lambda XT$ . In this paper we describe the sets of extended eigenvalues and extended eigenvectors for the quasi-normal operators.

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## 1. INTRODUCTION AND PRELIMINARIES

Let  $\mathcal{H}$  be a separable complex Hilbert space, and denote by  $\mathcal{B}(\mathcal{H})$  the algebra of all bounded linear operators on  $\mathcal{H}$ . If  $T$  is an operator in  $\mathcal{B}(\mathcal{H})$ , then a complex number  $\lambda$  is an extended eigenvalue of  $T$  if there is a nonzero operator  $X$  such that  $TX = \lambda XT$ . We denote by the symbol  $\sigma_{ext}(T)$  the set of extended eigenvalues of  $T$ . The subspace generated by extended eigenvectors corresponding to  $\lambda$  will be denoted by  $E_{ext}(T, \lambda)$ .

The concepts of extended eigenvalue and extended eigenvector are closely related with generalization of famous Lomonosov's theorem on existence of non-trivial hyperinvariant subspace for the compact operators on a Banach space, which were done by S. Brown in [4], and Kim, Moore and Pearcy in [9], and is stated as follows :

*If an operator  $T$  on a Banach space has a non-zero compact eigenvector, then  $T$  has a nontrivial hyperinvariant subspace.*

The special case, where  $T$  commutes with a non-zero compact operator, is the original theorem of Lomonosov [11].

Extended eigenvalues and their corresponding extended eigenvectors were studied by several authors (see for example [1], [2], [5], [8] and [10]).

In [2], Biswas, Lambert and Petrovic have introduced this notion and they have shown that  $\sigma_{ext}(V) = ]0, +\infty[$  where  $V$  is the well-known integral Volterra operator on the space  $L^2[0, 1]$ . In [8], Karaev gave a complete description of the set of extended eigenvectors of  $V$ .

Recently, in [12], Shkarin has shown that there is a compact quasinilpotent operator  $T$  for which  $\sigma_{ext}(T) = \{1\}$ , that which allows to classify this type of operators.

In [5], the authors has given an accurate and practical description of the set of extended eigenvectors of normal operators.

In this paper we treat a more generalize class of operators, that is the quasinormal operators.

In section 2 we introduce the sets of intertwining values of a couple of operators and  $\lambda$ -intertwining operators associated with a couple of operators and an intertwining value. We give a complete description of the set of intertwining values associated with a quasinormal operator and a operator of the form  $A \otimes S$  where  $A$  is an injective positive operator and  $S$  is the usual forward shift on the Hardy space  $H^2$ . This is the main result of the paper and it is used several times in the sequel. In particular, we apply this result in order to describe the extended spectrum of a pure quasinormal operator.

In section 3, Theorem 3.1 gives a description of extended eigenvectors for any injective subnormal operator. In particular, we give a description of extended eigenvectors related to the canonical decomposition of a subnormal operator in sum of normal and pure subnormal operators.

Section 4 is devoted to the complete description of the extended eigenvalues and the extended eigenspaces of a general quasinormal operator.

In section 5, we generalize a theorem of Yoshino which gives a necessary and sufficient condition that an operator commuting with a quasinormal operator have an extension commuting with the normal extension of the quasinormal operator. In particular, we generalize this to operators intertwining two quasinormal operators, such a result which will give us the relationship between extended eigenvectors of some pure quasinormal operators and their minimal normal extensions.

## 2. INTERTWINING VALUES AND $\lambda$ -INTERTWINING OPERATORS OF QUASI-NORMAL OPERATORS

In this section, we characterize the set of extended eigenvalues of a quasinormal operator. Recall that an operator  $T \in \mathcal{B}(\mathcal{H})$  is quasi-normal if it commutes with its modulus  $|T| := (T^*T)^{1/2}$ , i.e.,  $T|T| = |T|T$ . Furthermore,  $T$  is pure if it has no reducing subspaces  $\mathcal{M} \neq \{0\}$  such that  $T|_{\mathcal{M}}$  is normal. Since the normal operators have been accomplished in [5], we will focus in this section on the case of pure quasi-normal operators. First we will show some auxiliary results.

**Proposition 2.1.** *Let  $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ , then  $\sigma_{ext}(T_1)\sigma_{ext}(T_2) \subset \sigma_{ext}(T_1 \otimes T_2)$ , where  $T_1 \otimes T_2$  is the tensor product of  $T_1$  and  $T_2$ .*

*Proof.* Let  $\lambda_i \in \sigma_{ext}(T_i)$  and  $X_i \in E_{ext}(T_i, \lambda_i) \setminus \{0\}$ ,  $i = 1, 2$ . If we consider  $X := X_1 \otimes X_2$ , then  $X$  is a nonzero operator in  $E_{ext}(T_1 \otimes T_2, \lambda_1 \lambda_2)$ , which implies

$$\lambda_1 \lambda_2 \in \sigma_{ext}(T_1 \otimes T_2).$$

□

Now, if we denote by  $S$  the unilateral shift (which we suppose that it is acting on Hardy space  $H^2$ ), then A. Brown proved the following theorem (see [3]).

**Theorem 2.2.** *An operator  $T \in \mathcal{B}(\mathcal{H})$  is a pure quasinormal operator if and only if there is an injective positive operator  $A$  on a Hilbert space  $\mathfrak{L}$  such that  $T$  is unitarily equivalent to  $A \otimes S$ , acting on  $\mathfrak{L} \otimes H^2$ .*

**Remark 2.3.** The two following remarks will be frequently used in the sequel.

1) Let  $T = V_T|T| \in \mathcal{B}(\mathcal{H})$  be the polar decomposition of a pure quasinormal operator  $T$ . The subspace  $\mathfrak{L}_T = \mathcal{H} \ominus V_T\mathcal{H}$  is invariant by  $|T|$  and we can choose the positive operator  $A$  in the above theorem by setting  $A := A_T = |T|_{|\mathfrak{L}_T}$ . In this case we will denote by  $U_T \in \mathcal{B}(\mathcal{H}, \mathfrak{L}_T \otimes H^2)$  the unitary operator such that  $A_T \otimes S = U_T T U_T^*$ . Proposition 2.1 and Theorem 2.2 already show that  $\sigma_{ext}(A_T) \cdot \mathbb{D}^c \subseteq \sigma_{ext}(T)$ . We will frequently identify the space  $\mathfrak{L}_T \otimes H^2$  with the space  $\bigoplus_{k=0}^{\infty} \mathfrak{L}_T$ .

2) Let  $\mathcal{H}$  be a Hilbert space, and  $T \in \mathcal{B}(\mathcal{H})$ . Suppose that there exist  $X, U \in \mathcal{B}(\mathcal{H})$ , with  $U$  is an invertible operator, such that  $T = U^{-1} S U$ . Then, one can easily verify that  $\sigma_{ext}(T) = \sigma_{ext}(S)$ . Moreover, for all  $\lambda \in \sigma_{ext}(T)$ ,  $E_{ext}(T, \lambda) = U E_{ext}(S, \lambda) U^{-1}$ .

To our purpose, we introduce the following useful sets of operators. Let  $(A, B) \in \mathcal{B}(\mathcal{H}_1) \times \mathcal{B}(\mathcal{H}_2)$  and  $r \in \mathbb{R}_+^*$ , we define  $\mathcal{A}_r(A, B)$  by setting

$$\mathcal{A}_r(A, B) = \{L \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) : \exists c \geq 0 \text{ such that } \forall x \in \mathcal{H}_2, \\ \forall n \in \mathbb{N}, \|r^{-n} A^n L x\| \leq c \|B^n x\|\}.$$

When  $\mathcal{H}_1 = \mathcal{H}_2 := \mathcal{H}$  and  $A = B$ , the set  $\mathcal{A}_r(A, A)$  is denoted as  $\mathcal{A}_r(A)$  or if no confusion is possible, we write simply  $\mathcal{A}_r$ . Moreover, in the case of the positive operator  $A$  is invertible, the set  $\mathcal{A}_{|\lambda|}$  is defined by

$$\mathcal{A}_{|\lambda|} = \{L \in \mathcal{B}(\mathcal{H}) : \sup_{n \in \mathbb{N}} \|\lambda^{-n} A^n L A^{-n}\| < +\infty\}.$$

In addition, for  $|\lambda| = 1$ , we get the Deddens algebra given in [6]. We also define the intertwining values associated with the couple of operators  $(A, B)$  by setting  $\Lambda_{int}(A, B) = \{\lambda \in \mathbb{C} : \exists X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \text{ such that } AX = \lambda XB\}$ . If  $\lambda \in \Lambda_{int}(A, B)$ , we denote by  $E_{int}(A, B, \lambda)$  the space of  $\lambda$ -intertwining operators  $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ , that is operators such that  $AX = \lambda XB$ . When  $A = B$ ,  $\Lambda_{int}(A, A)$  is exactly the extended spectrum of the operator  $A$ , and if  $\lambda \in \sigma_{ext}(A)$ , then the space of  $\lambda$ -intertwining operators is exactly  $E_{ext}(T, \lambda)$ . The next result will be used several times in the sequel.

**Proposition 2.4.** *Let  $R$  be an injective quasinormal operator acting on a Hilbert space  $\mathcal{H}$  and let  $A$  be an injective positive operator on a Hilbert space  $\mathfrak{L}$ . Then,  $\lambda \in \Lambda_{int}(R, A \otimes S)$  if and only if  $\mathcal{A}_{|\lambda|}(R, A) \neq \{0\}$ .*

*Proof.* Assume that  $\lambda \in \Lambda_{int}(R, A \otimes S)$  and let  $X = [X_0, \dots, X_n, \dots] \in \mathcal{B}(\bigoplus_{k=0}^{+\infty} \mathfrak{L}, \mathcal{H})$  be a nonzero operator satisfying  $RX = \lambda X(A \otimes S)$ . Since  $R$  is injective, we have  $\lambda \neq 0$ . An easy calculation shows that  $RX_k = \lambda X_{k+1}A$ , and hence  $R^k X_0 = \lambda^k X_k A^k$  for any  $k \in \mathbb{N}$ . On the one hand, since the range of  $A^k$  is dense, it implies that  $X_0$  is necessarily non-null. On the other hand, we see that

$$|\lambda|^{-n} \|R^n X_0 x\| \leq \|X\| \|A^n x\|, \quad \forall n \in \mathbb{N}, \quad \forall x \in \mathfrak{L}.$$

Consequently,  $X_0$  is a non-zero element of  $\mathcal{A}_{|\lambda|}(R, A)$ .

Reciprocally, if  $L \in \mathcal{A}_{|\lambda|}(R, A) \setminus \{0\}$ , we define for all  $n \in \mathbb{N}$ , the operator

$$\begin{aligned} \check{X}_n &: \text{Im} A^n \rightarrow \mathcal{H} \\ A^n x &\mapsto \lambda^{-n} R^n L x. \end{aligned}$$

Since  $L \in \mathcal{A}_{|\lambda|}(R, A)$ , there is  $c \geq 0$  such that for all  $y \in \text{Im} A^n$ ,  $\|\check{X}_n y\| \leq c\|y\|$ . Also,  $\text{Im} A^n$  is dense in  $\mathfrak{L}$ , thus  $\check{X}_n$  has a (unique bounded) extension on  $\mathfrak{L}$ , which will be denoted by  $X_n$ . It remains to verify that  $RX_n = \lambda X_{n+1}A$ , for all  $n \in \mathbb{N}$ . So, let  $x \in \mathfrak{L}$ , and  $y = A^n x$ , then

$$\begin{aligned} RX_n y &= RX_n A^n x = R\check{X}_n A^n x = \lambda^{-n} R^{n+1} Lx \\ &= \lambda \lambda^{-(n+1)} R^{n+1} Lx = \lambda \check{X}_{n+1} A^{n+1} x = \lambda X_{n+1} Ay. \end{aligned}$$

By density, we get  $RX_n = \lambda X_{n+1}A$ , and hence  $X \in E_{\text{int}}(R, A \otimes S, \lambda) \setminus \{0\}$ , as we wanted.  $\square$

Let  $T$  be a self-adjoint operator acting on a Hilbert space  $H$ , we denote by  $m_T = \inf\{\langle Tx, x \rangle : \|x\| = 1\}$ . We observe that  $m_T = 1/\|T^{-1}\|$  when  $T$  is an invertible positive operator. Also denote, as usual, by  $\sigma(T)$  and  $\sigma_p(T)$  the spectrum and the point spectrum of  $T$  respectively. The following Theorem is the main result of the paper and will be used several times in the sequel.

**Theorem 2.5.** *Let  $R \in \mathcal{B}(\mathcal{H})$  be an injective quasinormal operator and let  $A \in \mathcal{B}(\mathfrak{L})$  be an injective positive operator, then we have:*

- (1) *if  $(m_{|R|}, \|A\|) \in \sigma_p(|R|) \times \sigma_p(A)$ , then*

$$\Lambda_{\text{int}}(R, A \otimes S) = \mathbb{D}(0, \frac{m_{|R|}}{\|A\|}]^c;$$

- (2) *if  $(m_{|R|}, \|A\|) \notin \sigma_p(|R|) \times \sigma_p(A)$ , then*

$$\Lambda_{\text{int}}(R, A \otimes S) = \mathbb{D}(0, \frac{m_{|R|}}{\|A\|}]^c.$$

*Proof.* The first step consists in proving the inclusion

$$\mathbb{D}(0, \frac{m_{|R|}}{\|A\|}]^c \subseteq \Lambda_{\text{int}}(R, A \otimes S).$$

Let  $0 < \varepsilon < \|A\|$ , if we denote by  $E^A$  (resp.  $E^{|R|}$ ) the spectral measure of  $A$  (resp.  $|R|$ ), then we can choose a nonzero vector  $a$  (resp. a nonzero vector  $b$ ) in  $E^{|R|}([m_{|R|}, m_{|R|} + \varepsilon])(\mathcal{H})$  (resp. in  $E^A([\|A\| - \varepsilon, \|A\|])(\mathfrak{L})$ ) because  $m_{|R|} = \inf\{\lambda : \lambda \in \sigma(|R|)\}$  (resp.  $\|A\| = \sup\{\lambda : \lambda \in \sigma(A)\}$ ). Observe that  $b$  can be written under the form  $b = A^n b_n$  where

$$b_n = \left( \int_{\|A\| - \varepsilon}^{\|A\|} t^{-n} dE^A(t) \right) b.$$

Set  $L = a \otimes b$ , since  $R$  is quasinormal we have

$$\begin{aligned} \|R^n Lx\| &= \| |R|^n a \| \langle x, b \rangle \| = \| |R|^n a \| \|b_n\| \langle A^n x, \frac{b_n}{\|b_n\|} \rangle \\ &\leq \| |R|^n a \| \|b_n\| \|A^n x\| \leq \|a\| \|b\| \left( \frac{m_{|R|} + \varepsilon}{\|A\| - \varepsilon} \right)^n \|A^n x\|. \end{aligned}$$

Hence, the non-null operator  $L$  belongs to  $\mathcal{A}_{\frac{m_{|R|} + \varepsilon}{\|A\| - \varepsilon}}(R, A)$ . Applying

Proposition 2.4, we see that  $\mathbb{D}(0, \frac{m_{|R|} + \varepsilon}{\|A\| - \varepsilon}]^c \subseteq \Lambda_{\text{int}}(R, A \otimes S)$ . Since  $\varepsilon$  could be arbitrarily chosen in  $]0, \|A\|]$ , we obtained the desired inclusion.

The second step is to prove that  $\Lambda_{\text{int}}(R, A \otimes S) \subseteq \mathbb{D}(0, \frac{m_{|R|}}{\|A\|}]^c$ . Let  $\lambda \in \Lambda_{\text{int}}(R, A \otimes S)$ , we know from Proposition 2.4 that there exists a non-null

operator  $L \in \mathcal{A}_{|\lambda|}(R, A)$ . Recall that since the operator  $R$  is quasinormal, we have  $|R^n| = |R|^n$ . Therefore, there exists an absolute positive constant  $C$  such that  $(m_{|R|})^{2n} L^* L \leq L^* |R|^{2n} L = L^* R^{*n} R^n L \leq C^2 |\lambda|^{2n} A^{2n}$ . Then, we necessarily have

$$\frac{m_{|R|}}{\|A\|} \left( \frac{\|L\|}{C} \right)^{\frac{1}{n}} \leq |\lambda|,$$

and letting  $n \rightarrow \infty$  we obtained the desired conclusion. We are now in position to prove the announced assertions.

(1) By hypothesis, there exists a couple of unit eigenvectors  $(u, v) \in \mathcal{H} \times \mathcal{L}$  such that  $|R|u = m_{|R|}u$  and  $Av = \|A\|v$ . We can see that the operator  $L = u \otimes v$  is in  $\mathcal{A}_{m_{|R|}/\|A\|}(R, A)$ . From Proposition 2.4, we deduced that the circle  $\mathcal{C}(0, m_{|R|}/\|A\|)$  centered in 0 and of radius  $m_{|R|}/\|A\|$  is contained in  $\Lambda_{int}(R, A \otimes S)$ . Using the two firsts steps of the proof, we can conclude.

(2) Suppose that  $\lambda \in \Lambda_{int}(R, A \otimes S)$  with  $|\lambda| = \frac{m_{|R|}}{\|A\|}$ , then there exists  $X \neq 0$  such that  $RX = \lambda X(A \otimes S)$ . Since  $R$  is injective, then  $\lambda \neq 0$  and hence  $|R|$  is invertible ( $m_{|R|} > 0$ ). As in the proof of the last proposition, we write  $X = [X_0, \dots, X_n, \dots]$ , and we get  $R^n X_0 = \lambda^n X_n A^n$ . Let  $R = V|R|$  be the polar decomposition of the operator  $R$ , since  $R$  is injective we see that  $V$  is an isometry. Choose  $x \in \mathcal{L}$  and  $y = R^{*n}b \in Im(R^{*n})$ , we derive that

$$| \langle X_0 x, y \rangle | = \left( \frac{m_{|R|}}{\|A\|} \right)^n | \langle X_n A^n x, b \rangle | \leq \|X\| \left( \frac{A}{\|A\|} \right)^n x \|m_{|R|}^n \|b\|.$$

Since  $R$  is quasinormal, the isometry  $V$  commute with  $|R|$ . Then, observe that we can choose  $b$  in the closure of the range of  $R^n$  which is contained in the range of  $V^n$ , hence we can write  $b = V^n c$ . Therefore, we get  $\| |R|^{-n} y \| = \| V^{*n} b \| = \| V^{*n} V^n c \| = \| c \| = \| b \|$ . Then, using the density of the range of  $R^{*n}$ , for all  $(x, y) \in \mathcal{L} \times \mathcal{H}$  we obtain

$$| \langle X_0 x, y \rangle | \leq \|X\| \left( \frac{A}{\|A\|} \right)^n x \|m_{|R|}^n |R|^{-n} y\|.$$

But

$$\|m_{|R|}^n |R|^{-n} y\|^2 = \int_{m_{|R|}}^{\|R\|} m_{|R|}^{2n} \frac{1}{t^{2n}} dE_{y,y}^{[R]}(t) \xrightarrow{n \rightarrow +\infty} E_{y,y}^{[R]}(\{m_{|R|}\}).$$

Similarly, we see

$$\left\| \left( \frac{A}{\|A\|} \right)^n x \right\|^2 \xrightarrow{n \rightarrow +\infty} E_{x,x}^A(\{\|A\|\}).$$

According to the assumptions of (2), we must have at least one of the two spectral projections  $E^{[R]}(\{m_{|R|}\})$  or  $E^A(\{\|A\|\})$  null. Thus, using the three previous facts, we see that  $X_0 = 0$  which implies  $X = 0$ . So, we get a contradiction. Consequently, using Proposition 2.4, it follows that the circle  $\mathcal{C}(0, m_{|R|}/\|A\|)$  does not intersect  $\Lambda_{int}(R, A \otimes S)$ . From the two firsts steps of the proof, we derive that  $\Lambda_{int}(R, A \otimes S) = \mathbb{D}(0, \frac{m_{|T|}}{\|T\|}]^c$ . This finishes the proof of Theorem 2.5.  $\square$

**Corollary 2.6.** *Let  $T$  be a pure quasinormal operator acting on a Hilbert space  $\mathcal{H}$ . We have  $\sigma_{\text{ext}}(T) = \mathbb{D}(0, \frac{m_{|T|}}{\|T\|}]^c$  when  $m_{|T|}$  and  $\|T\|$  are in  $\sigma_p(|T|)$ , and we have  $\sigma_{\text{ext}}(T) = \mathbb{D}(0, \frac{m_{|T|}}{\|T\|}]^c$  when  $m_{|T|}$  and  $\|T\|$  are not both in  $\sigma_p(|T|)$ .*

*Proof.* Applying Theorem 2.2 and taking into account Remark 2.3 we see that  $T$  is unitarily equivalent to the operator  $A_T \otimes S$  acting on the Hilbert space  $\mathfrak{L}_T \otimes H^2$ , where  $\mathfrak{L}_T = \mathcal{H} \ominus V_T \mathcal{H}$ . We set for simplicity  $A := A_T$ . We clearly have  $m_{|T|} = m_A$ ,  $\|T\| = \|A\|$  and  $\sigma_p(|T|) = \sigma_p(A)$ . Therefore, from now on, we may assume that  $T$  is under the form  $A \otimes S$  and  $\mathcal{H} = \mathfrak{L}_T \otimes H^2$ . Then, it suffices to apply Theorem 2.5 with  $R = A \otimes S$ .  $\square$

### 3. CASE OF SUBNORMAL OPERATORS

It is known that an operator  $T \in \mathcal{B}(\mathcal{H})$  is subnormal if there is a Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}$  and a normal operator  $N \in \mathcal{B}(\mathcal{K})$  such that  $S = N|_{\mathcal{H}}$ . This extension is minimal (m.n.e.) if  $\mathcal{K}$  has no proper subspace that reduces  $N$  and contains  $\mathcal{H}$ . In addition, we know that every quasinormal operator is subnormal. So, we show the following theorem in the more general case, that is the subnormal one. In particular, it is true for quasinormal operators.

**Theorem 3.1.** *Let  $N \in \mathcal{B}(\mathcal{E})$  and  $T \in \mathcal{B}(\mathcal{F})$  be normal and pure subnormal operators respectively, such that the operator  $R = N \oplus T \in \mathcal{B}(\mathcal{E} \oplus \mathcal{F})$  is injective. Let  $\lambda \in \sigma_{\text{ext}}(Z)$ , and let*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in E_{\text{ext}}(Z, \lambda).$$

*Then  $X_3 = 0$ ,  $X_1 \in E_{\text{ext}}(N, \lambda)$ ,  $X_4 \in E_{\text{ext}}(T, \lambda)$  and  $X_2 \in E_{\text{int}}(N, T, \lambda)$ .*

*Proof.* The hypothesis imply

$$(3.1) \quad \begin{cases} NX_1 = \lambda X_1 N \\ NX_2 = \lambda X_2 T \\ TX_3 = \lambda X_3 N \\ TX_4 = \lambda X_4 T \end{cases}$$

Clearly, it suffices to show that  $X_3 = 0$ . So, let

$$M = \begin{bmatrix} T & Y \\ 0 & T_1 \end{bmatrix} \in \mathcal{B}(\mathcal{F} \oplus \mathcal{G}),$$

be the m.n.e. of  $T$ , and consider the following operators defined on  $\mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}$  by

$$\tilde{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & T & Y \\ 0 & 0 & T_1 \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} 0 & 0 & 0 \\ X_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then formulas 3.1 imply  $\tilde{M}\tilde{X} = \lambda\tilde{X}\tilde{N}$ . But both  $\tilde{M}$  and  $\tilde{N}$  are normal operators, by using the Fuglede-Putnam theorem it follows  $\tilde{M}^*\tilde{X} = \bar{\lambda}\tilde{X}\tilde{N}^*$ . From this, we have easily  $T^*X_3 = \bar{\lambda}X_3N^*$ . Hence, for all  $m, n \in \mathbb{N}$  we get the following system

$$\begin{cases} T^n X_3 = \lambda^n X_3 N^n \\ T^{*m} X_3 = \bar{\lambda}^m X_3 N^{*m}, \end{cases}$$

which implies, since  $N$  is normal

$$T^{*m}T^nX_3 = T^nT^{*m}X_3.$$

Consequently

$$(T^{*m}T^n - T^nT^{*m})X_3 = 0, \quad \forall m, n \in \mathbb{N},$$

which means

$$\text{Im}(X_3) \subset \bigcap_{m,n \in \mathbb{N}} \ker(T^{*m}T^n - T^nT^{*m}) := \mathcal{M}.$$

Now, let  $x \in \mathcal{M}$  then

$$T^{*m}T^n(Tx) = T^{*m}T^{n+1}x = T^{n+1}T^{*m}x = T^nT^{*m}(Tx).$$

Thus  $\mathcal{M} \in \mathcal{Lat}(T)$ . Furthermore, if  $x \in \mathcal{M}$  then

$$T^{*m}T^n(T^*x) = T^{*m}(T^nT^*x) = T^{*m+1}T^n x = T^nT^{*m}(T^*x).$$

Hence  $\mathcal{M} \in \mathcal{Lat}(T^*)$ . From this  $\mathcal{M}$  is a reducing subspace for  $T$ . Therefore, there are two operators  $M_1 \in \mathcal{B}(\mathcal{M})$ ,  $M_1 \in \mathcal{B}(\mathcal{M}^\perp)$  such that  $T = M_1 \oplus M_2$ . Moreover, in  $\mathcal{M} \oplus \mathcal{M}^\perp$  the operators  $TT^*$  and  $T^*T$  have the following representations

$$TT^* = M_1M_1^* \oplus M_2M_2^*, \quad T^*T = M_1^*M_1 \oplus M_2^*M_2.$$

Finally, let  $x \in \mathcal{M}$ , then  $TT^*x = T^*Tx$  which implies  $M_1M_1^*x = M_1^*M_1x$ . So  $M_1$  is normal, and we get  $\mathcal{M} = 0$  since  $T$  is pure. Consequently  $X_3 = 0$  and the proof is complete.  $\square$

#### 4. EXTENDED EIGENVALUES AND EXTENDED EIGENSPACES OF QUASI-NORMAL OPERATORS

The following theorem describes the spaces of extended eigenvectors of a pure quasi normal operator. We will use the notations introduced in Remark 2.3.

**Theorem 4.1.** *Let  $T$  be a pure quasinormal operator acting on a Hilbert space  $\mathcal{H}$ . Let  $\lambda \in \sigma_{ext}(T)$  then*

$$E_{ext}(T, \lambda) = \text{weak}^*\text{-span}\{U_T^*(I \otimes S^m)\text{diag}(L, X_{1,1}, \dots, X_{n,n}, \dots)U_T : m \in \mathbb{N}, L \in \mathcal{A}_{|\lambda|}(A_T)\},$$

where  $X_{n,n}$  is, for all  $n \in \mathbb{N}$ , the (unique bounded) extension on  $\mathfrak{L}_T$  of the operator

$$\begin{aligned} \check{X}_{n,n} : \text{Im}A_T^n &\rightarrow \mathfrak{L}_T \\ A_T^n x &\mapsto \lambda^{-n} A_T^n Lx. \end{aligned}$$

*Proof.* As usual, we set  $A := A_T$ ,  $\mathfrak{L}_T = \mathcal{H} \ominus V_T \mathcal{H}$  and  $\mathcal{A}_{|\lambda|} := \mathcal{A}_{|\lambda|}(A_T)$ . Let  $X_0 \in \mathcal{B}(\mathcal{H})$  be a nonzero solution of  $TX_0 = \lambda X_0 T$ . Then, we have seen that  $X_0 = U_T^* X U_T$  where  $X \in \mathcal{B}(\mathfrak{L}_T \otimes H^2)$  is solution of the equation  $(A \otimes S)X = \lambda X(A \otimes S)$ . Let  $(A_{i,j})_{i,j \geq 0}$  be the matrix of  $A \otimes S$  in  $\mathfrak{L} \otimes H^2$ , i.e.,

$$A_{i,j} = \begin{cases} A & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases}$$



Consider for all  $\alpha \in \overline{\mathbb{D}}$ , the operator  $J_\alpha$  whose the matrix in  $\mathfrak{L} \otimes H^2$  is defined by

$$(J_\alpha)_{i,j} = \begin{cases} \alpha^i I & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then one can verify that  $J_\alpha(A \otimes S) = \alpha(A \otimes S)J_\alpha$ . In particular,  $J_0(A \otimes S) = 0$ . Now let  $\lambda \in \sigma_{ext}(T)$  and let  $X = (X_{i,j})$  be a nonzero operator acting on  $\mathfrak{L} \otimes H^2$ , and  $\lambda \in \mathbb{C}$  (necessarily nonzero) verifying  $(A \otimes S)X = \lambda X(A \otimes S)$ . A left composition by  $J_0$  gives

$$0 = J_0(A \otimes S)X = \lambda J_0X(A \otimes S) = J_0X(A \otimes S).$$

But

$$(J_0X(A \otimes S))_{i,j} = \begin{cases} X_{0,j+1}A & \text{if } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$

which implies  $X_{0,j+1} = 0$  for all  $j$ , since  $A$  has dense range. In addition, we know that  $(A \otimes S)X = \lambda X(A \otimes S)$  implies  $(A \otimes S)^n X = \lambda^n X(A \otimes S)^n$  for all  $n \in \mathbb{N}$ . A same process gives  $X_{n,m} = 0$  for all  $0 < n < m$ . Consequently,  $X$  has a lower triangular matrix.

Thus, if we denote by  $X(m)$  the operator whose the matrix is

$$(X(m))_{i,j} = \begin{cases} X_{i,j} & \text{if } j = m + i \\ 0 & \text{otherwise,} \end{cases}$$

for all  $m \in \mathbb{Z}$ . We can prove that

$$X = \text{weak}^* \lim_{n \rightarrow +\infty} \left( \sum_{k=0}^n \left( 1 - \frac{k}{n+1} \right) X(-k) \right).$$

Moreover, we observe that there exists an operator  $Y$  acting on  $\mathfrak{L} \otimes H^2$  such that  $X(-n) = (I \otimes S^n)(Y(0))$  for all  $n \in \mathbb{N}$ . Furthermore, one can verify that  $(A \otimes S)X(-n) = \lambda X(-n)(A \otimes S)$  if and only if  $(A \otimes S)Y(0) = \lambda Y(0)(A \otimes S)$ . Therefore, we are reduced to examine the case where  $X = X(0)$ .

We have

$$((A \otimes S)X(0))_{i,j} = \begin{cases} AX_{i-1,i-1} & \text{if } i = j + 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\lambda X(0)(A \otimes S))_{i,j} = \begin{cases} \lambda X_{i,i}A & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for all  $n \in \mathbb{N}$  we have  $AX_{n,n} = \lambda X_{n+1,n+1}A$ . Thus, we get for all  $n$ ,  $\lambda^{-n}A^n X_{0,0} = X_{n,n}A^n$ . On the one hand, since the range of  $A$  is dense, it implies that  $X_{0,0}$  is necessarily non-null. On the other hand, we see that

$$|\lambda|^{-n} \|A^n X_{0,0}x\| \leq \|X\| \|A^n x\|, \quad \forall n \in \mathbb{N}, \quad \forall x \in \mathfrak{L}.$$

Consequently,  $X_{0,0}$  is a non-zero element of  $\mathcal{A}_{|\lambda|} := \mathcal{A}_{|\lambda|}(A)$ .

Reciprocally, if  $L \in \mathcal{A}_{|\lambda|} \setminus \{0\}$ , we define for all  $n \in \mathbb{N}$ , the operator

$$\begin{aligned} \check{X}_{n,n} &: \text{Im} A^n \rightarrow \mathfrak{L} \\ A^n x &\mapsto \lambda^{-n} A^n Lx. \end{aligned}$$

Since  $L \in \mathcal{A}_{|\lambda|}$ , there is  $c \geq 0$  such that for all  $y \in \text{Im} A^n$ ,  $\|\check{X}_{n,n}y\| \leq c\|y\|$ . Also,  $\text{Im} A^n$  is dense in  $\mathfrak{L}$ , thus  $\check{X}_{n,n}$  has a (unique bounded) extension

on  $\mathfrak{L}$ , which will be denoted by  $X_{n,n}$ . It remains to verify that  $AX_{n,n} = \lambda X_{n+1,n+1}A$ , for all  $n \in \mathbb{N}$ . So, let  $x \in \mathfrak{L}$ , and  $y = A^n x$ , then

$$\begin{aligned} AX_{n,n}y &= AX_{n,n}A^n x = A\check{X}_{n,n}A^n x = \lambda^{-n}A^{n+1}Lx \\ &= \lambda\lambda^{-n-1}A^{n+1}Lx = \lambda\check{X}_{n+1,n+1}A^{n+1}x = \lambda X_{n+1,n+1}Ay, \end{aligned}$$

by density, we get  $AX_{n,n} = \lambda X_{n+1,n+1}A$ , which implies  $X \in E_{ext}(T, \lambda) \setminus \{0\}$ , as we wanted.  $\square$

**Remark 4.2.** Let  $A$  be an injective positive operator on a Hilbert space  $\mathfrak{L}$  and let  $|\lambda| \leq 1$ , then we can easily verify that  $\mathcal{A}_{|\lambda|} := \mathcal{A}_{|\lambda|}(A)$  is an algebra, that which is not true in general, when  $|\lambda| > 1$  (see Example (2)). Recall that  $\mathcal{A}_1$  is the Deddens algebra given in [6]. Finally, If we denote by  $D_{A,L,\lambda}$  the operator defined on  $\mathfrak{L} \otimes H^2$  by

$$(D_{A,L,\lambda})_{i,j \in \mathbb{N}} = \begin{cases} \lambda^{-i}A^i L A^{-i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then, we get the following corollary.

**Corollary 4.3.** *Let  $A$  be an invertible positive operator on a Hilbert space  $\mathfrak{L}$ , and  $T = A \otimes S$ . If  $\lambda \in \sigma_{ext}(T)$  then*

$$E_{ext}(T, \lambda) = \text{weak}^*\text{-span}\{(I \otimes S^n)D_{A,L,\lambda} : n \in \mathbb{N}, L \in \mathcal{A}_{|\lambda|}(A)\}.$$

**Example 1.** Let  $T = A \otimes S$  such that

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \alpha > \beta > 0,$$

and let

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

Then, if  $\lambda \in \mathbb{C}^*$  we get

$$\lambda^{-n}A^n L A^{-n} = \begin{bmatrix} \lambda^{-n}a & (\frac{\alpha}{\lambda\beta})^n b \\ (\frac{\beta}{\lambda\alpha})^n c & \lambda^{-n}d \end{bmatrix}.$$

So we distinguish the following cases :

- (1) if  $|\lambda| \geq \frac{\alpha}{\beta} > 1$ , then  $\mathcal{A}_{|\lambda|} = \mathcal{B}(\mathfrak{L}) = \mathcal{M}_2(\mathbb{C})$ .
- (2) if  $1 \leq |\lambda| < \frac{\alpha}{\beta}$ , then

$$\mathcal{A}_{|\lambda|} = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}, \quad a, c, d \in \mathbb{C} \right\}.$$

- (3) if  $\frac{\beta}{\alpha} \leq |\lambda| < 1$ , then

$$\mathcal{A}_{|\lambda|} = \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, \quad c \in \mathbb{C} \right\}.$$

- (4) if  $|\lambda| < \frac{\beta}{\alpha}$ , then  $\mathcal{A}_{|\lambda|} = \{0\}$ .

**Example 2.** Let  $T = A \otimes S$  such that

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \alpha > \beta > \gamma > 0,$$

and let

$$L_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, one can verify that

$$\|\lambda^{-n} A^n L_1 A^{-n}\| = |\lambda|^{-n} \left(\frac{\alpha}{\beta}\right)^n,$$

$$\|\lambda^{-n} A^n L_2 A^{-n}\| = |\lambda|^{-n} \left(\frac{\beta}{\gamma}\right)^n,$$

and

$$\|\lambda^{-n} A^n L_1 L_2 A^{-n}\| = |\lambda|^{-n} \left(\frac{\alpha}{\gamma}\right)^n.$$

Now, let  $\alpha, \beta, \gamma$  and  $\lambda$  be such that

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = |\lambda|,$$

then, clearly  $L_1, L_2 \in \mathcal{A}_{|\lambda|}$ , but  $L_1 L_2 \notin \mathcal{A}_{|\lambda|}$ .

In the next result, we describe completely the extended eigenspaces of a general quasinormal operator.

**Theorem 4.4.** *Let  $R$  be an injective quasinormal operator on a Hilbert space  $\tilde{\mathcal{H}}$ , and consider  $R = N \oplus T \in \mathcal{B}(\mathcal{E} \oplus \mathcal{H})$  the canonical decomposition of  $R$  into a direct sum of a normal operator  $N \in \mathcal{B}(\mathcal{E})$  and a pure quasinormal operator  $T \in \mathcal{B}(\mathcal{H})$ . Let  $\lambda \in \sigma_{\text{ext}}(R)$  then  $E_{\text{ext}}(R, \lambda)$  is the following subspace of  $\mathcal{B}(\mathcal{E} \oplus \mathcal{H})$*

$$\left\{ \begin{bmatrix} U & VU_T \\ 0 & W \end{bmatrix} : U \in E_{\text{ext}}(N, \lambda), V \in E_{\text{int}}(N, A_T \otimes S, \lambda), W \in E_{\text{ext}}(T, \lambda) \right\}$$

where  $E_{\text{int}}(N, A_T \otimes S, \lambda)$  is the set of operators  $V \in \mathcal{B}(\oplus_{k=0}^{+\infty} \mathfrak{L}_T, \mathcal{E})$  whose matricial form are given by  $V = [V_0, \dots, V_n, \dots]$ , where  $V_0 \in \mathcal{A}_{|\lambda|}(N, A_T)$  and  $V_n$  is, for all  $n \in \mathbb{N}^*$ , the (unique bounded) extension on  $\mathfrak{L}$  of the operator

$$\begin{array}{ccc} \check{V}_n & : & \text{Im} A^n \rightarrow \mathcal{E} \\ & & A^n x \mapsto \lambda^{-n} N^n V_0 x. \end{array}$$

*Proof.* Let  $X$  be an extended eigenvector of  $R$  associated with the extended eigenvalue  $\lambda$ . According to Theorem 3.1 it suffices to describe the upper off-diagonal coefficient  $X_2$  in the matrix representation of  $X$  with respect to the orthogonal direct sum  $\tilde{\mathcal{H}} = \mathcal{E} \oplus \mathcal{H}$ . Clearly, we have  $X_2 = VU_T$  where  $V = [V_0, \dots, V_n, \dots] \in E_{\text{int}}(N, A_T \otimes S, \lambda)$ . For convenience, we write  $A = A_T$ . We see that  $N^n V_0 = \lambda^n V_n A^n$  for every  $n \in \mathbb{N}$ . Thus, we have  $\|N^n V_0 x\| \leq \|V\| |\lambda|^n \|A^n x\|$  and hence  $V_0 \in \mathcal{A}_{|\lambda|}(N, A)$ .

Conversely, by using assumptions, we get  $NV = \lambda A \otimes S$  and any matrix of the form

$$X = \begin{bmatrix} U & VU_T \\ 0 & W \end{bmatrix},$$

where  $U \in E_{\text{ext}}(N, \lambda)$  and  $W \in E_{\text{ext}}(T, \lambda)$ , is an extended eigenvector of  $R$ . It ends the proof.  $\square$

We can now describe the extended spectrum of a general quasinormal operator.

**Corollary 4.5.** *Let  $R$ ,  $N$ ,  $T$  and  $\tilde{\mathcal{H}}$  be as in the last theorem, then we have*

$$\sigma_{ext}(R) = \sigma_{ext}(N \oplus T) = \sigma_{ext}(N) \cup \mathbb{D}(0, \frac{m_{|N|} \wedge m_{|T|}}{\|T\|}]^c$$

*if one of the following assumptions holds:*

- a)  $m_{|N|} < m_{|T|}$  and  $(m_{|N|}, \|T\|) \in \sigma_p(|N|) \times \sigma_p(|T|)$ ;
- b)  $m_{|N|} = m_{|T|}$ , and  $(m_{|N|}, \|T\|) \in \sigma_p(|N|) \times \sigma_p(T)$  or  $(m_{|T|}, \|T\|) \in \sigma_p(|T|)^2$ ;
- c)  $m_{|N|} > m_{|T|}$  and  $(m_{|T|}, \|T\|) \in \sigma_p(|T|)^2$ .

*Else we have*

$$\sigma_{ext}(R) = \sigma_{ext}(N) \cup \mathbb{D}(0, \frac{m_{|N|} \wedge m_{|T|}}{\|T\|}]^c.$$

*Proof.* Using Theorem 4.4, we see that

$$\sigma_{ext}(R) = \sigma_{ext}(N) \cup \sigma_{ext}(T) \cup \Lambda_{int}(N, A_T \otimes S).$$

Taking into account Corollary 2.6, we see that the proof rests on an application of Theorem 2.5.  $\square$

## 5. LIFTING OF EIGENVECTORS OF PURE QUASI-NORMAL OPERATORS

In [13, Theorems 1 and 3], the author gives a necessary and sufficient condition that an operator commuting with a quasinormal operator have an extension commuting with the normal extension of the quasinormal operator. In Theorem 5.3 we generalize this to operators intertwining two quasinormal operators. First we introduce the following proposition (see [7] for the proof).

**Proposition 5.1.** *Let  $T_i \in \mathcal{B}(\mathcal{H}_i)$  be subnormal operator with m.n.e.  $N_i \in \mathcal{B}(\mathcal{K}_i)$ ,  $i = 1, 2$ , and let  $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ . The following are equivalent :*

- (1)  *$X$  has a (unique) extension  $\hat{X} \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$  such that  $N_1 \hat{X} = \hat{X} N_2$ .*
- (2) *There exists a constant  $c \geq 0$  such that*

$$\sum_{i,j=0}^n \langle T_1^i X x_j, T_1^j X x_i \rangle \leq c \sum_{i,j=0}^n \langle T_2^i x_j, T_2^j x_i \rangle$$

*for all finite set  $\{x_0, \dots, x_n\}$  in  $\mathcal{H}_2$ .*

*Moreover, if (2) holds, then  $T_1 X = X T_2$ .*

We also need the following auxiliary lemma (see [13] for the proof).

**Lemma 5.2.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be injective quasinormal operator with the polar decomposition  $T = V|T|$ , and let  $N \in \mathcal{B}(\mathcal{K})$  be the m.n.e. of  $T$  with the polar decomposition  $N = U|N|$ . Then  $U$  is unitary and*

$$V = U|_{\mathcal{H}} \text{ and } |T| = |N|_{\mathcal{H}}.$$

Now we prove the main theorem of this section.

**Theorem 5.3.** *Let  $T_i \in \mathcal{B}(\mathcal{H}_i)$  be injective quasinormal operator with the polar decomposition  $T_i = V_i|T_i|$  and let  $N_i \in \mathcal{B}(\mathcal{K}_i)$ , be the m.n.e. of  $T_i$  with the polar decomposition  $N_i = U_i|N_i|$ ,  $i = 1, 2$ . If  $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ , then the following are equivalent :*

- (1)  $X$  has a (unique) extension  $\hat{X} \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$  such that  $N_1\hat{X} = \hat{X}N_2$ .
- (2)  $V_1X = XV_2$  and  $|T_1|X = X|T_2|$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $\hat{X} \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$  be an extension of  $X$  such that  $N_1\hat{X} = \hat{X}N_2$ . Then by using Fuglede-Putnam theorem, we get  $N_1^*\hat{X} = \hat{X}N_2^*$ . Hence, we easily get

$$|N_1|\hat{X} = \hat{X}|N_2| \text{ and } U_1\hat{X} = \hat{X}U_2.$$

By Lemma 5.2, for all  $x \in \mathcal{H}_2$

$$V_1Xx = V_1\hat{X}x = U_1\hat{X}x = \hat{X}U_2x = \hat{X}V_2x = XV_2x,$$

and

$$|T_1|Xx = |T_1|\hat{X}x = |N_1|\hat{X}x = \hat{X}|N_2|x = \hat{X}|T_2|x = X|T_2|x.$$

(2)  $\Rightarrow$  (1). Let  $U_i \in \mathcal{B}(\mathcal{K}'_i)$  be the minimal unitary extension of  $V_i$ . Then for any finite set  $\{x_0, \dots, x_n\}$  in  $\mathcal{H}_2$

$$\begin{aligned} \sum_{i,j=0}^n \langle T_1^i X x_j, T_1^j X x_i \rangle &= \sum_{i,j=0}^n \langle V_1^i |T_1|^i X x_j, V_1^j |T_1|^j X x_i \rangle \\ &= \sum_{i,j=0}^n \langle V_1^i X |T_2|^j x_j, V_1^j X |T_2|^i x_i \rangle = \sum_{i,j=0}^n \langle U_1^i X |T_2|^j x_j, U_1^j X |T_2|^i x_i \rangle \\ &= \sum_{i,j=0}^n \langle U_1^{*j} X |T_2|^j x_j, U_1^{*i} X |T_2|^i x_i \rangle = \left\| \sum_{i=0}^n U_1^{*i} X |T_2|^i x_i \right\|_{\mathcal{K}'_1}^2. \end{aligned}$$

Since for all  $k \geq 0$

$$\begin{aligned} U_1^{*i} X |T_2|^i x_i &= U_1^{*i+k} U_1^k X |T_2|^i x_i = U_1^{*i+k} X V_2^k |T_2|^i x_i \\ &= U_1^{*i+k} X U_2^k |T_2|^i x_i = U_1^{*i+k} X U_2^{i+k} U_2^{*i} |T_2|^i x_i, \end{aligned}$$

for all  $i = 0, \dots, n$ , we have, by choosing  $k$  such that  $i + k = n$  for each  $i$

$$\begin{aligned} \left\| \sum_{i=0}^n U_1^{*i} X |T_2|^i x_i \right\|_{\mathcal{K}'_1}^2 &= \left\| U_1^{*n} X U_2^n \sum_{i=0}^n U_2^{*i} |T_2|^i x_i \right\|_{\mathcal{K}'_1}^2 \\ &= \left\| X U_2^n \sum_{i=0}^n U_2^{*i} |T_2|^i x_i \right\|_{\mathcal{K}'_1}^2 = \left\| X U_2^n \sum_{i=0}^n U_2^{*i} |T_2|^i x_i \right\|_{\mathcal{H}_1}^2 \\ &\leq \|X\|_{\mathcal{H}_1}^2 \left\| \sum_{i=0}^n U_2^{*i} |T_2|^i x_i \right\|_{\mathcal{K}'_1}^2 = \|X\|_{\mathcal{H}_1}^2 \sum_{i,j=0}^n \langle U_2^i |T_2|^j x_j, U_2^j |T_2|^i x_i \rangle \\ &= \|X\|_{\mathcal{H}_1}^2 \sum_{i,j=0}^n \langle |T_2|^j V_2^i x_j, |T_2|^i V_2^j x_i \rangle = \|X\|_{\mathcal{H}_1}^2 \sum_{i,j=0}^n \langle T_2^j x_j, T_2^i x_i \rangle \end{aligned}$$

Hence, Proposition 5.1 implies the first assertion. The proof is now complete.  $\square$

Now, let  $A$  be a positive operator, and denote by  $U$  the bilateral shift, then  $A \otimes U$  is the m.n.e. of  $A \otimes S$ , and the last theorem implies the following corollary.

**Corollary 5.4.** *Let  $A$  be an injective positive operator on a Hilbert space  $\mathfrak{L}$ , and let  $X$  be a bounded operator on  $\mathfrak{L} \otimes H^2$ , then the following are equivalent*

- (1)  $X$  has a (unique) extension  $\hat{X} \in \mathcal{B}(\mathfrak{L} \otimes L^2)$  such that  
 $(A \otimes U)\hat{X} = \lambda\hat{X}(A \otimes U).$
- (2)  $(A \otimes I)X = |\lambda|X(A \otimes I)$  and  $(I \otimes S)X = \lambda/|\lambda|X(I \otimes S).$

Now, we use similar arguments from the proofs of Theorem 2.5 and Theorem 4.1 to establish the following result.

**Theorem 5.5.** *Let  $A$  be an invertible positive operator on a Hilbert space  $\mathfrak{L}$ , and  $N = A \otimes U$ . Denote by  $a := \|A\| \|A^{-1}\|$ , then if  $(\|A\|, \|A^{-1}\|^{-1}) \in \sigma_p(A)^2$  we have*

$$\sigma_{ext}(N) = \{z \in \mathbb{C} : \frac{1}{a} \leq |z| \leq a\},$$

else we have

$$\sigma_{ext}(N) = \{z \in \mathbb{C} : \frac{1}{a} < |z| < a\}.$$

Moreover, if  $\lambda \in \sigma_{ext}(N)$  then

$$E_{ext}(N, \lambda) = \text{weak}^*\text{-span}\{(I \otimes U^m)\hat{D}_{A,L,\lambda} : m \in \mathbb{Z}, L \in \hat{\mathcal{A}}_{|\lambda|}\}.$$

where  $\hat{D}_{A,L,\lambda}$  is the operator defined by

$$(\hat{D}_{A,L,\lambda})_{i,j \in \mathbb{Z}} = \begin{cases} \lambda^{-i} A^i L A^{-i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\hat{\mathcal{A}}_{|\lambda|} = \{L \in \mathcal{B}(\mathfrak{L}) : \sup_{n \in \mathbb{Z}} \|\lambda^{-n} A^n L A^{-n}\| < +\infty\}.$$

Indeed, if  $(\|A\|, \|A^{-1}\|^{-1}) \in \sigma_p(A)^2$  it suffices to consider  $L_1 = u \otimes v$  and  $L_2 = v \otimes u$  where  $u$  and  $v$  are eigenvectors of  $A$  associated with  $\|A\|$  and  $\|A^{-1}\|^{-1}$  respectively.

Else, we use the inequality

$$|\langle Lx, y \rangle| \leq C \left\| \left( \frac{A}{\|A\|} \right)^m x \right\| \left\| \left( \frac{A^{-1}}{\|A^{-1}\|} \right)^m y \right\|$$

(which is available for any  $L \in \hat{\mathcal{A}}_a$  and any  $m \in \mathbb{N}$ ), in order to show that  $L$  is necessarily null (see proof of Theorem 2.5). We proceed similarly for proving that  $\hat{\mathcal{A}}_{a^{-1}} = \{0\}$ .

**Theorem 5.6.** *Let  $A$  be an invertible positive operator on a Hilbert space  $\mathfrak{L}$ ,  $T = A \otimes S$  and  $N = A \otimes U$ . Let  $\lambda \in \sigma_{ext}(N)$  and  $X \in \mathcal{B}(\mathfrak{L} \otimes H^2)$ , then  $X$  has a (unique) extension  $\hat{X} \in \mathcal{B}(\mathfrak{L} \otimes L^2)$  such that*

$$N\hat{X} = \lambda\hat{X}N,$$

if and only if

$$X \in \text{weak}^*\text{-span}\{(I \otimes S^n)D_{A,L,\lambda} : n \in \mathbb{N}, L \in E_{ext}(A, |\lambda|)\}.$$

*Proof.* Let  $\hat{X} = (\hat{X}_{i,j})_{i,j \in \mathbb{Z}}$  be an operator acting on  $\mathfrak{L} \otimes L^2$  such that

$$(5.1) \quad N\hat{X} = \lambda\hat{X}N,$$

Consider for all  $\alpha \in \mathbb{T}$ , the operator  $\hat{J}_\alpha$  whose the matrix in  $\mathfrak{L} \otimes L^2$  is defined by

$$(\hat{J}_\alpha)_{i,j} = \begin{cases} \alpha^i I & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then one can verify that  $\hat{J}_\alpha N = \alpha N \hat{J}_\alpha$ . Hence, let  $\alpha, \beta \in \mathbb{T}$ , if we apply to both sides of the Equation (5.1) the operator  $\hat{J}_\alpha$  from the left, and the operator  $\hat{J}_\beta$  from the right, we get

$$N\hat{J}_\alpha\hat{X}\hat{J}_\beta = \frac{\lambda}{\alpha\beta}\hat{J}_\alpha\hat{X}\hat{J}_\beta N.$$

Now, let  $m \in \mathbb{Z}$ ,  $\theta \in [0, 2\pi]$  and put  $\alpha = \beta^{-1} = e^{i\theta}$ , then the last equation implies

$$N \int_0^{2\pi} e^{-im\theta} \hat{J}_{e^{i\theta}} \hat{X} \hat{J}_{e^{-i\theta}} dm(\theta) = \lambda \int_0^{2\pi} e^{-im\theta} \hat{J}_{e^{i\theta}} \hat{X} \hat{J}_{e^{-i\theta}} dm(\theta) N,$$

where the integrals are well defined in Bochner sense. Hence

$$N\hat{X}(m) = \lambda\hat{X}(m)N.$$

where  $\hat{X}(m)$  is the operator acting on  $\mathfrak{L} \otimes L^2$  whose the matrix is given by

$$(\hat{X}(m))_{i,j} = \begin{cases} \hat{X}_{i,j} & \text{if } i = m + j \\ 0 & \text{otherwise,} \end{cases}$$

On the other hand, one can easily verify that  $\hat{J}_{e^{i\theta}} \hat{X} \hat{J}_{e^{-i\theta}}$  is an extension of the operator  $J_{e^{i\theta}} X J_{e^{-i\theta}}$ , so that  $\hat{X}(m)$  is an extension of  $X(m) = (I \otimes S^m) D_{A,L,\lambda}$ . Also, by using the last theorem, there exists  $L \in \hat{A}_{|\lambda|}$  such that

$$\hat{X}(m) = (I \otimes U^m) \hat{D}_{A,L,\lambda}.$$

Now, suppose that  $m < 0$ . In this case,  $L = 0$ . Indeed, if  $L \neq 0$ , then  $\mathfrak{L} \otimes H^2 \notin \mathcal{Lat}(\hat{X}(m))$ , which means that there is no bounded operator on  $\mathfrak{L} \otimes H^2$  for which  $\hat{X}(m)$  is an extension. Now assume that  $m \geq 0$ . Then  $\hat{X}(m)$  is an extension of the operator  $X(m)$ , and by using the Corollary 5.4, we have an equivalence with the two following equations

$$(A \otimes I)X = |\lambda|X(A \otimes I) \quad \text{and} \quad (I \otimes S)X = \lambda/|\lambda|X(I \otimes S).$$

One can easily verify that the last equalities are equivalent to

$$AL = |\lambda|LA.$$

which means that  $L \in E_{ext}(A, |\lambda|)$ . The converse is easy and will be left to the reader.  $\square$

**Remark 5.7.** Let  $A$  be an invertible positive operator on a Hilbert space  $\mathfrak{L}$  such that  $(\|A\|, \|A^{-1}\|^{-1}) \in \sigma_p(A)^2$ ,  $T = A \otimes S$  and  $N = A \otimes U$  the m.n.e. of  $T$ . As a direct result of the last theorem, we can summarize the relationship between extended eigenvectors of  $T$  and  $N$  in the four following

cases :

1) If  $|\lambda| \in [1/a, a]$  and let

$$X = (I \otimes S^n)D_{A,L,\lambda}, \quad n \in \mathbb{N}.$$

Suppose that  $L \in E_{\text{ext}}(A, |\lambda|)$ , then  $X$  has a (unique) extension  $\hat{X} \in E_{\text{ext}}(N, \lambda)$ .

2) With the same hypotheses, if we suppose that  $L \notin E_{\text{ext}}(A, |\lambda|)$ , then  $X$  doesn't have any extension in  $E_{\text{ext}}(N, \lambda)$ .

3) Let  $|\lambda| \in [1/a, a]$  and  $\hat{X} \in E_{\text{ext}}(N, \lambda) \setminus \{0\}$  be such that

$$\hat{X} = (I \otimes U^m)\hat{D}_{A,L,\lambda}, \quad m < 0,$$

then there is no bounded operator on  $\mathfrak{L} \otimes H^2$  for which  $\hat{X}$  is an extension.

4) If  $|\lambda| > a$ , and let

$$X = (I \otimes S^n)D_{A,L,\lambda}, \quad n \in \mathbb{N}, \quad L \in \mathcal{A}_{|\lambda|} \setminus \{0\},$$

then  $X \in E_{\text{ext}}(T, \lambda)$ , but it has no extension in  $E_{\text{ext}}(N, \lambda)$ .

When  $(\|A\|, \|A^{-1}\|^{-1}) \notin \sigma_p(A)^2$ , the reader can adapt this remark by using Theorem 5.5 and conclude.

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